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A characterization of nuclei in orthomodular and quantic lattices

L. Román and B. Rumbos

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Area de la Investigación Cientifica, Circuito Exterior, Ciudad Universitaria, CP 04510, Mexico

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Abstract

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We prove that nuclei in orthomodular lattices, and more generally in quantic lattices are completely determined by the central elements.

Introduction

The purpose of this paper is to give some new results concerning nuclei in orthomodular lattices and more generally in quantic lattices. Nuclei have proved important since, in particular, they correspond with quotients or surjections of certain lattices such as Boolean algebras and locales (see [6], for example). As is well known, nuclei have their main source in the theory of topoi and sheafification due to Grothendieck, Lawvere and Tierney (see [4, 8] for details).

In Section 1 we deal with the classic notion of a nucleus on a lattice and apply this to orthomodular lattices, characterizing such nuclei.

Section 2 talks about our generalization of several new kinds of 'conjunctions' in lattices (perhaps the best name is products in lattices), see [13] for more details, we define nuclei in these lattices and show that they coincide with the nuclei defined in Section 1.

1. The characterization of nuclei in orthomodular lattices

Throughout this work, L will denote an *orthomodular lattice*, that is, a lattice $(L, \wedge, \vee, 0, 1)$ with an orthocomplement $\perp : L \to L$ satisfying:

(i) $a \leq b \Rightarrow b^{\perp} \leq a^{\perp}$,
(ii) $a^{\perp} \vee a = 1$,
(iii) $a^{\perp} \wedge a = 0$,
for all a,b in L. Moreover, L satisfies the following *weak modularity property*:

$$a \leq b \quad \Leftrightarrow \quad b = a \vee (a^{\perp} \wedge b).$$

These lattices have traditionally been associated with 'quantum logic', since the propositions for a quantum system correspond with closed subspaces of a Hilbert space and these constitute an orthomodular lattice.

As pointed out in [12] a natural choice for an implication conjunction pair is to take the *Sasaki arrow* defined by

$$a \rightarrow b = (a \wedge b) \vee a^{\perp} \quad \text{for all } a,b \in L$$

together with the conjunction '&' given as

$$a \mathbin{\&} b = (a \vee b^{\perp}) \wedge b \quad \text{for all } a,b \in L.$$

The relation $a \mathbin{\&} b \leq c$ iff $a \leq b \rightarrow c$ holds for any elements a,b,c of L. In categorical terms, if L is seen as a category, then this is just saying that the functor $- \mathbin{\&} b$ is a left adjoint to $b \rightarrow -$ for any $b \in L$. An immediate result from this is that the operator & distributes arbitrary joins (on the left), i.e. it is 'left distributive'.

Some obvious consequences of the definition of '&' are:
(i) $a \wedge b \leq a \mathbin{\&} b$,
(ii) $a \mathbin{\&} b \leq b$,
(iii) $1 \mathbin{\&} a = a = a \mathbin{\&} 1$,
(iv) $a \mathbin{\&} 0 = 0 = 0 \mathbin{\&} a$,
(v) $a \mathbin{\&} a = a$.

It is interesting to note that the conjunction '&' is not necessarily commutative or associative, actually '&' possesses any one of these two properties, if and only if L is a Boolean algebra. (That associativity implies L Boolean is not entirely trivial. The reader is referred to [14] for a proof.)

It is clear that for any $a \in L$ the orthocomplement of a can be written as $a \rightarrow 0$ and satisfies the following property:

$$x \mathbin{\&} b = 0 \quad \Leftrightarrow \quad x \leq b^{\perp}.$$

We recall the definition of a nucleus:

Definition 1.1. Let L denote a suplattice, then a function $j : L \rightarrow L$ is called a *nucleus* if and only if the following are satisfied:

(i) $a \le j(a)$ (j is inflationary),
(ii) $j \circ j(a) = j(a)$ (j is idempotent),
(iii) $j(a \wedge b) = j(a) \wedge j(b)$ (j is $\wedge$-preserving).

If L is any Heyting algebra (respectively Boolean algebra), then for any $a \in L$ the operator defined by $u_b(x) = b \vee x$ is easily shown to be a nucleus. Notice that in order for u_b to be a nucleus we must have a *distributive lattice*. In [9], a detailed study of nuclei in Heyting algebras is given. In particular, if L denotes a Boolean algebra, the following is true:

Proposition 1.2 (Macnab [9]). *If $j: L \to L$ is a nucleus over a Boolean algebra, then $j(a) = u_{j(0)}$.* □

We are interested in giving a characterization of nuclei over orthomodular lattices. Since an orthomodular lattice is not necessarily distributive (unless it is a Boolean algebra), the problem seems to be different from the Boolean case. However, we shall see that this is not the case. Indeed, a similar result (as in Boolean algebras) will hold. We begin with the following:

Definition 1.3. Given L and L' two complete orthomodular lattices, a function $f: L \to L'$ is said to be a morphism of orthomodular lattices iff the following properties hold:
(i) $f(\bigvee_i a_i) = \bigvee_i f(a_i)$, for every family $\{a_i\} \subseteq L$,
(ii) $f(a \wedge b) = f(a) \wedge f(b)$, for all $a, b \in L$,
(iii) $f(1) = 1$,
(iv) $f(a^\perp) = f(a)^\perp$, for all $a \in L$.

As usual, f is a surjection whenever the image of f is L'. Given $f: L \to L'$, a surjection between orthomodular lattices, f is in particular a surjection of suplattices and so by the adjoint functor theorem f has a right adjoint $f^*: L' \to L$. It is fairly straightforward to check that the function defined by the composition $f^* \circ f: L \to L$ is indeed a nucleus.

Given a nucleus $j: L \to L$, the image of L under j will be denoted by L_j. The map induced by j between these two suplattices is clearly a surjection and it will turn out, as we shall see later that it is indeed a surjection of orthomodular lattices.

In order to give our characterization of nuclei, we note that the real problem here is that we do not have a distributive lattice. As when we have a group which is not necessarily abelian, a natural step is to take the *center* of an orthomodular lattice. We then have the following:

Definition 1.4. If L is an arbitrary orthomodular lattice, then the center of L, denoted by $Z(L)$ is the set

$$\{x \in L \mid x \,\&\, a = a \,\&\, x \;\forall a \in L\}\,.$$

Clearly, $Z(L)$ is nonempty (at least 0 and 1 are in $Z(L)$) and is a Boolean algebra with the operations induced by L. Whenever we have a complete orthomodular lattice, $Z(L)$ is a complete Boolean algebra. Moreover, the next lemma can be proved easily.

Lemma 1.5. *If L is a (complete) orthomodular lattice and z is an arbitrary element of $Z(L)$, then the operator $u_z : L \to L$ is a nucleus.* □

The only thing to be noted is that as soon as you have an element of the center of L, then the *distributive law* holds, that is, given x, y arbitrary elements of L and z an element of $Z(L)$, then:

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)\,.$$

What about the converse to Lemma 1.5? We shall see that it is true.

Lemma 1.6. *Let $j : L \to L$ be a nucleus on L, then $j(x^{\perp}) \,\&\, j(x) = j(0)$.*

Proof. Since $x^{\perp} \wedge x = 0$, we have the following identities:

$$j(0) = j(x^{\perp} \wedge x) = j(x^{\perp}) \wedge j(x) \leq j(x^{\perp}) \,\&\, j(x)\,.$$

Conversely, by the definition of & we have that:

$$j(x^{\perp}) \,\&\, j(x) = [\, j(x^{\perp}) \vee j(x)^{\perp}] \wedge j(x)\,,$$

and since j is inflationary and $\perp$ reverses order we get:

$$\begin{aligned} j(x^{\perp}) \,\&\, j(x) &\leq (\, j(x^{\perp}) \vee x^{\perp}) \wedge j(x) \\ &= j(x^{\perp}) \wedge j(x) = j(x^{\perp} \wedge x) = j(0)\,, \end{aligned}$$

yielding the desired result. □

In [3] it is proved that & satisfies the following property (we will call it the 'absorption' identity): Let $x, y, z \in L$ with $x \leq y$, then $(z \,\&\, y) \,\&\, x = z \,\&\, x$. We shall make use of this in the next lemma.

Lemma 1.7. *If $j : L \to L$ is a nucleus on L, then $j(0) \in Z(L)$.*

Proof. Let $x \in L$, then we already know that $j(0) \wedge x \leq j(0) \,\&\, x$. So we must only check the converse inequality. Again, by the definition of & and Lemma 1.6, we have:

$$j(0) \,\&\, x = (j(x^\perp) \,\&\, j(x)) \,\&\, x\,.$$

Now, since j is inflationary and by the absorption identity the right-hand side is equal to $j(x^\perp) \,\&\, x$. Using the definition of & we have now:

$$j(x^\perp) \,\&\, x = [\,j(x^\perp) \vee x^\perp] \wedge x$$

and again, j inflationary implies the left-hand side is equal to $j(x^\perp) \wedge x$ which clearly is less than or equal to $j(0)$. Finally since $j(0) \,\&\, x \le x$ we have the desired result. Therefore, $j(0) \,\&\, x = j(0) \wedge x$, yielding $j(0) \in Z(L)$. □

Using these last two lemmas we can now prove the next theorem:

Theorem 1.8. *Let $j: L \to L$ be an arbitrary function on L. Then j is a nucleus if and only if $j(x) = u_z(x)$ for any $x \in L$, where z is an element of the center of L.*

Proof. We only have to prove that if j is a nucleus, then it has the desired form. Clearly, (by Lemma 1.7) our best candidate is $u_{j(0)}$. Moreover, for any $x \in L$ we have $x \vee j(0) \le j(x)$.

We will show that $j(x) \,\&\, x^\perp \le x^\perp \to j(0)$. Because $x^\perp \to j(0)$ is equal to $x \vee j(0)$, recall that $j(0)$ is an element of $Z(L)$. We calculate $j(x) \,\&\, x^\perp$:

$$\begin{aligned} j(x) \,\&\, x^\perp &= [\,j(x) \vee x] \wedge x^\perp \\ &= j(x) \wedge x^\perp \le j(x) \wedge j(x^\perp) = j(0)\,. \end{aligned}$$

So by adjointness we get the desired result, and the proof of the theorem is complete. □

We shall see that L_j is indeed orthomodular. Recall that weak modularity can be alternatively expressed by the following identity:

$$a \le b \quad \Leftrightarrow \quad = b \wedge (b^\perp \vee a)\,.$$

Corollary 1.9. *Let $j: L \to L$ be a nucleus, then $j: L \to L_j$ is a surjection of orthomodular lattices.*

Proof. L_j is the complete lattice $[\,j(0), 1]$ and given $j(x) \in L_j$ the orthocomplement of this element in L_j is clearly $j(x^\perp) = x^\perp \vee j(0)$. Indeed,

$$j(x) \vee j(x^{\perp}) = 1 .$$

$$\begin{aligned} j(x) \wedge j(x^{\perp}) &= (x \vee j(0)) \wedge j(x^{\perp}) \\ &= x \wedge j(x^{\perp}) \wedge j(0) \wedge j(x^{\perp}) \\ &\leq j(x) \wedge j(x^{\perp}) \wedge j(0) \wedge j(x^{\perp}) \\ &= j(0) \wedge j(0) = j(0) . \end{aligned}$$

So, we only have to check that weak modularity is satisfied:

Let $j(x), j(y) \in L_j$ with $j(x) \leq j(y)$, then using weak orthomodularity in L we have:

$$j(x) = j(y) \wedge (j(y)^{\perp} \vee j(x)) .$$

Now, by Theorem 1.8, $j(x) = x \vee j(0)$. Hence replacing this value we get:

$$j(x) = [(x \vee j(0)) \vee (y^{\perp} \wedge j(0)^{\perp})] \wedge j(y) .$$

Finally, since $j(y)$ is an element of $Z(L)$ the left-hand side can be written as:

$$[(x \vee j(0)) \vee (y^{\perp} \vee j(0))] \wedge j(y) .$$

which is equal to $j(y) \wedge (j(x) \vee j(y)^{\perp})$ and the result follows. So L_j is indeed an orthomodular lattice and the morphism $j : L \to L_j$ is then trivially a surjection of orthomodular lattices. □

2. Quantic nuclei

Recently the study of several 'product' operations in a complete lattice L appeared in the literature (see [1, 10] for details). Moreover, Niefeld and Rosenthal in [11] study the concept of quantic nucleus over lattices that they call 'Quantales'. Such lattices were first defined by Mulvey and Borceux. In [13], we introduced the concept of 'Quantic lattice' and quantic nuclei on such lattices. This category contains both orthomodular lattices and quantales as its objects. One might ask if these quantic nuclei differ from the usual nuclei. We will see that this is not the case. We introduce first the definition of a quantic lattice and a quantic nucleus.

Definition 2.1. Let Q be a complete lattice $(Q, \leq)$, then Q is a *quantic lattice* iff there exists a binary operation $\& : Q \times Q \to Q$ satisfying:

(i) $-\& \, q : Q \to Q$ is a poset morphism for all q in Q,

(ii) $-\& \, q : Q \to Q$ has a right adjoint $q \to -$ for all q in Q.

The reader can find some interesting examples of such lattices in [14]. Here, we mention some of them:

(1) Any locale with $a \& b = a \wedge b$, $\forall a, b \in L$.

(2) Any complete orthomodular lattice, where $\&$ is the operation already defined in Section 1.

(3) Any quantale (see [1] for details).

Quantic nuclei were first introduced by [2]. Our definition does not differ from theirs. The definition is as follows:

Definition 2.2. Let Q be any quantic lattice. If $j: Q \to Q$ is an order preserving function then j is a *quantic nucleus* if and only if the following are satisfied:

(i) $a \le j(a)$,

(ii) $j^2 a = a$,

(iii) $j(a) \& j(b) \le j(a \& b)$,

for all a,b in Q.

Example. If z is an arbitrary element of the center of an orthomodular lattice, then clearly the function $u_z: L \to L$ already defined in Section 2 is a quantic nucleus.

Notice that apparently this definition seems broader than the definition of nucleus. For some time we tried to find an example of a nonobvious quantic nucleus over an orthomodular lattice and we never succeeded. The reason is very simple, the only quantic nuclei over an orthomodular lattice are the usual nuclei. In order to prove this assertion we begin with the following lemma:

Lemma 2.3. *If $j: L \to L$ is a quantic nucleus, then $j(0)$ is an element of the center of L.*

Proof. Since $j(a) \& j(b) \le j(a \& b)$, then taking $b = a^\perp$ we get:

$$j(a) \& j(a^\perp) \le j(a \& a^\perp) = j(0) = j(a) \wedge j(a^\perp) .$$

This last equality can be proved as follows:

$$j(a) \wedge j(a^\perp) \le j(a) \& j(a^\perp) \le j(a \& a^\perp) = j(0)$$

and

$$j(0) = j(a \wedge a^\perp) \le j(a) \wedge j(a^\perp) .$$

Moreover, an easy calculation shows that $j(a) \& j(a^\perp) = j(a) \wedge j(a^\perp)$. Therefore, $j(a) \& j(a^\perp) = j(0)$.

Now, taking any element $a \in L$, we calculate $j(0) \,\&\, a^{\perp}$:

$$j(0) \,\&\, a^{\perp} = [\,j(a) \,\&\, j(a^{\perp})] \,\&\, a^{\perp} = j(a) \,\&\, a^{\perp}$$

Because $a^{\perp} \leq j(a^{\perp})$ and by the absorption identity (see Section 1). Now,

$$\begin{aligned} j(a) \,\&\, a^{\perp} &= [\,j(a) \vee a] \wedge a \\ &= j(a) \wedge a^{\perp} \leq j(a) \wedge j(a^{\perp}) = j(0)\,. \end{aligned}$$

Since we always have $j(a) \,\&\, a^{\perp} \leq a^{\perp}$ we get $j(0) \,\&\, a^{\perp} = j(0) \wedge a^{\perp}$. In particular, $j(0) \,\&\, a = j(0) \wedge a$; this means that $j(0)$ is an element of $Z(L)$. □

Finally we have the following theorem:

Theorem 2.4. *Let L be an orthomodular lattice. If $j: L \rightarrow L$ is an arbitrary function, then j is a quantic nucleus iff $j = u_z$, where z is an arbitrary element of the center of L.*

Proof. We only need to show that if $j: L \rightarrow L$ is a quantic nucleus, then j has the desired form. By Lemma 2.3, our natural candidate again is $j(0)$. We must check only that $j(a) \leq a \vee j(a)$.

We shall use a similar proof as in Theorem 1.8. We know that $a^{\perp} \rightarrow j(0) = a \vee j(0)$ since $j(0)$ is an element of the center of L. We calculate $j(a) \,\&\, a^{\perp}$:

$$j(a) \,\&\, a^{\perp} = [\,j(a) \vee a] \wedge a^{\perp}$$

and the left-hand side is equal to $j(a) \wedge a^{\perp} \leq j(a) \wedge j(a^{\perp}) = j(0)$, by Lemma 2.3. So by adjointness we get $j(a) \leq a^{\perp} \rightarrow j(0)$ and the result follows. □

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